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A periodic phase soliton of the ultradiscrete hungry Lotka–Volterra equation

Shinya Nakamura

Major in Pure and Applied Mathematics, Graduate School of Fundamental Science and Engineering, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

E-mail: s-nakamura@moegi.waseda.jp

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Abstract

We propose a new type of solution to the ultradiscrete hungry Lotka–Volterra (uhLV) equation. For the solution, the periodic phase is introduced into the known soliton and the extended soliton becomes a traveling wave showing a periodic variation. We call this type of wave a 'periodic phase soliton' (PPS). The solution has two forms of expression: one is the 'perturbation form' and the other is the 'ultradiscrete permanent form'. We analyze the interaction among PPSs and solitons. Moreover, we give the outline of proof to show that the solution satisfies the bilinear equation of the uhLV equation.

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1. Introduction

The hungry Lotka–Volterra (uhLV) equation is a generalization of the Lotka–Volterra equation and is a predator–prey model of generalized interaction of species [1]. It is defined by

$$\frac{\mathrm{d}x_n}{\mathrm{d}t} = x_n\left(\sum_{k=1}^{M} x_{n-k} - \sum_{k=1}^{M} x_{n+k}\right), \tag{1}$$

where $x_n(t)$ denotes the number of nth species at time t and M means a range of interaction of nth species. This equation is a soliton equation and has the N-soliton solution. The discrete analog to (1) is the discrete hungry Lotka–Volterra equation defined by

$$\frac{x_n^{m+1}}{x_n^m} = \prod_{k=1}^{M} \frac{1+\delta x_{n-k}^m}{1+\delta x_{n+k}^{m+1}}, \tag{2}$$

where m denotes an integer time [4]. If we replace x_n^m by $x_n(m\delta)$ and take a limit $\delta \to 0$, we obtain (1) from (2) as a limit equation. Equation (2) is also a soliton equation and can be bilinearlized using a transformation of the dependent variable,

$$x_n^m = \frac{f_{n+M+1}^{m+1} f_{n-M}^m}{f_{n+1}^{m+1} f_n^m}. \tag{3}$$

Using this transformation, we obtain the bilinear equation

$$(1+\delta) f_{n+1}^m f_n^{m+1} = f_n^m f_{n+1}^{m+1} + \delta f_{n-M}^m f_{n+M+1}^{m+1}. \tag{4}$$

The ultradiscretization [2, 5, 7] is a process to discretize the dependent variable of a discrete equation utilizing the formula

$$\lim_{\varepsilon\to+0} \varepsilon \log(\mathrm{e}^{a/\varepsilon} + \mathrm{e}^{b/\varepsilon}) = \max(a, b). \tag{5}$$

By this process, we can obtain the 'ultradiscrete equation' of which all variables are discrete. In particular for the soliton equation, its integrability is preserved through the ultradiscretization. Thus, we obtain the ultradiscrete soliton equation through the process. For example, we obtain the set of equations and transformation from (2), (3) and (4) through the transformation of variables and constants including the parameter ε,

$$x_n^m \to \mathrm{e}^{x_n^m/\varepsilon}, \qquad f_n^m \to \mathrm{e}^{f_n^m/\varepsilon}, \qquad \delta \to \mathrm{e}^{-1/\varepsilon}, \tag{6}$$

and the limit $\varepsilon \to +0$. The obtained equations are

$$x_n^{m+1} - x_n^m = \sum_{k=1}^{M} \max\left(0, x_{n-k}^m - 1\right) - \sum_{k=1}^{M} \max\left(0, x_{n+k}^{m+1} - 1\right), \tag{7}$$

$$x_n^m = f_{n-M}^m + f_{n+M+1}^{m+1} - f_n^m - f_{n+1}^{m+1}, \tag{8}$$

$$f_{n+1}^m + f_n^{m+1} = \max\left(f_n^m + f_{n+1}^{m+1},\ f_{n-M}^m + f_{n+M+1}^{m+1} - 1\right). \tag{9}$$

We refer to (7) as the uhLV equation. The N-soliton solution to (4) can also be ultradiscretized giving those to (9). The solution in the 'perturbation form' to (4) is given by [4]

$$f_n^m = \sum_{\mu\in\{0,1\}^N} \prod_{1\leqslant i\leqslant N} s_i(m,n)^{\mu_i} \cdot \prod_{1\leqslant i<j\leqslant N} a_{ij}^{\mu_i\mu_j} \tag{10}$$

where

$$\begin{aligned} &s_i(m,n) = \omega_i^m k_i^{-n} c_i, \qquad \omega_i = \frac{1+\delta p_i}{1+\delta q_i}, \qquad a_{ij} = \frac{(p_i-p_j)(q_i-q_j)}{(p_i-q_j)(q_i-p_j)}, \\ &p_i = \sum_{j=0}^{M} k_i^j, \qquad q_i = \sum_{j=0}^{M} k_i^{-j}, \qquad \mu = \{\mu_1, \ldots, \mu_N\} \in \{0,1\}^N. \end{aligned} \tag{11}$$

The parameters k_i and c_i are arbitrary. If we use the following transformations with (6):

$$k_i \to \mathrm{e}^{k_i/\varepsilon}, \qquad c_i \to \mathrm{e}^{c_i/\varepsilon}, \tag{12}$$

and take the limit $\varepsilon \to +0$ under the condition $k_i \geqslant 1/M$, we obtain the ultradiscrete solution to (9),

$$f_n^m = \max_{\mu\in\{0,1\}^N} \left(\sum_{1\leqslant i\leqslant N} \mu_i s_i(m,n) + \sum_{1\leqslant i<j\leqslant N} \mu_i\mu_j a_{ij} \right), \tag{13}$$

where

$$\begin{aligned} &s_i(m,n) = \omega_i m - k_i n + c_i, \qquad \omega_i = M k_i - 1, \\ &a_{ij} = -(M+1)\min(k_i, k_j). \end{aligned} \tag{14}$$

The parameters k_i and c_i are also arbitrary other than $k_i \geqslant 1/M$. Note that we use formula (5) in the derivation of this solution.

The solution in the 'determinant form' to (4) is given by

$$f_n^m = \det(\varphi_i(m, n + j - 1))_{1\leqslant i,\, j\leqslant N}, \tag{15}$$

where

$$\varphi_i(m, n) = \alpha_i(1 + \delta p_i)^{-m}(1 - 1/p_i)^n + \beta_i(1 + \delta q_i)^{-m}(1 - 1/q_i)^n. \tag{16}$$

The parameters p_i and q_i are the same as in (11), and α_i and β_i are arbitrary. This determinant form cannot be ultradiscretized directly since the limit $\lim_{\varepsilon\to+0} \varepsilon \log(\mathrm{e}^{a/\varepsilon} - \mathrm{e}^{b/\varepsilon})$ is not well defined.

The contents of this paper are as follows. In section 2, we propose a new type of solution to (9) for $M = 2$. The special case of this solution gives a normal soliton. The general case gives a wave like a soliton including a periodic variation of its phase. Each wave moves changing its shape periodically due to this variation. We call this type of soliton a 'periodic phase soliton' (PPS) in this paper.

Moreover, the solution can be described by two equivalent forms. One is the perturbation form like (13). The other is the 'ultradiscrete permanent' (UP) form. The UP is obtained by ultradiscretizing the permanent and this type of solution can be considered to be an ultradiscrete analog to the determinant type of solution like (15) to the discrete equation [3, 6].

In section 3, we show concrete examples of a PPS. We pick up one and two PPSs or the interaction between one PPS and one soliton, and show their time evolution in detail.

In section 4, we give the outline of proof showing the PPS solution satisfying the bilinear equation (9). First we show the equivalence of the perturbation form and the UP form [3]. Then we show that the PPS solution of UP form satisfies the bilinear equation (9).

In section 5, we give concluding remarks.

2. Periodic phase soliton

We give an exact PPS solution to (9) for $M = 2$ in this section. The bilinear equation is

$$f_{n+1}^m + f_n^{m+1} = \max\left(f_n^m + f_{n+1}^{m+1},\ f_{n-2}^m + f_{n+3}^{m+1} - 1\right). \tag{17}$$

There are two forms to express the solution. One is the perturbation form and the other is the UP form. The solution in the perturbation form is given by

$$f_n^m = \max_{\mu_i\in\{0,1\}^N}\left(\sum_{i=1}^N \mu_i s_i(m, n) - 3\sum_{i=1}^N \mu_i k_i \sum_{j=1}^{i-1}\mu_j + \sum_{i=1}^N \mu_i p_i\left(n + \sum_{j=1}^{i-1}\mu_j\right)\right), \tag{18}$$

where $s_i(m, n) = \omega_i m - k_i n + c_i$ and $\omega_i = 2k_i - 1$ for $M = 2$. Note that we assume $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_N$ ($\geqslant 1/2$) without loss of generality for the solution and the term $\sum_{1\leqslant i<j\leqslant N}\mu_i\mu_j a_{ij}$ in (13) becomes $-3\sum_{i=1}^N \mu_i k_i \sum_{j=1}^{i-1}\mu_j$ in (18) under this assumption. Thus, the first two terms in the max function of (18) are derived from (13).

The feature of this solution is the last term in the max function, $\sum_{i=1}^N \mu_i p_i\left(n + \sum_{j=1}^{i-1}\mu_j\right)$. If $p_i \equiv 0$ for any i, the solution becomes the normal N-soliton solution equivalent to (13) for $M = 2$. The function $p_i(n)$ is periodic on n with period 2, that is,

$$p_i(n) = p_i(n + 2). \tag{19}$$

Moreover, the following additional condition is necessary for k_i:

$$\begin{aligned} &k_i \geqslant |p_i(n) - p_i(n + 1)|, \\ &|k_i - k_j| \geqslant |p_i(n) - p_i(n + 1) - (p_j(n) - p_j(n + 1))|. \end{aligned} \tag{20}$$

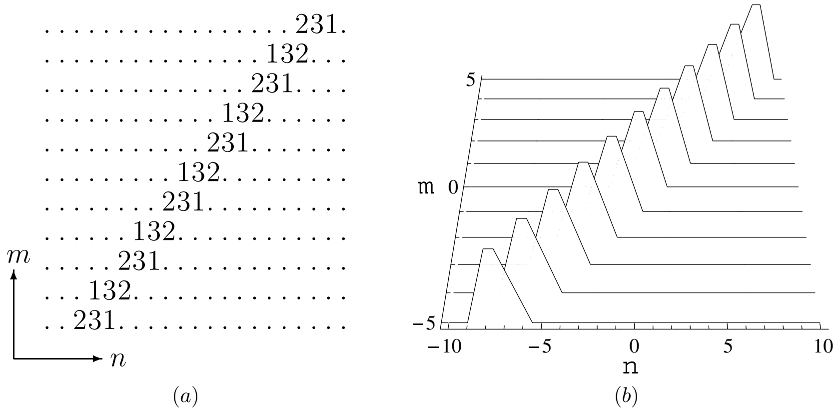

Figure 1. Plot of the 1-soliton solution for $N = 1$, $k_1 = 2$ ($\omega_1 = 3$), $c_1 = 0$ and $p_1 \equiv 0$. (*a*) Values on lattice points, (*b*) continuous profile for n.

We introduce the ultradiscrete permanent before giving the solution in the UP form. The permanent is a sign-free determinant defined by

$$\mathrm{perm}(a_{ij}) = \sum_{\pi} a_{1\pi_1} a_{2\pi_2} \ldots a_{N\pi_N} \tag{21}$$

where (a_{ij}) is $N \times N$ matrix and $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ is any possible permutation of numbers $(1, 2, \ldots, N)$. If we use the transformation $a_{ij} \to \mathrm{e}^{a_{ij}/\varepsilon}$ and take the limit $\varepsilon \to +0$, we obtain

$$\lim_{\varepsilon \to +0} \varepsilon \log \mathrm{perm}(\mathrm{e}^{a_{ij}/\varepsilon}) = \max_{\pi}(a_{1\pi_1} + a_{2\pi_2} + \cdots + a_{N\pi_N}). \tag{22}$$

Since we can consider the above expression as the max operation for the matrix (a_{ij}), we refer to it as $\max(a_{ij})$ and we call it the ultradiscrete permanent.

The solution to (17) in the UP form is given by

$$f_n^m = \tfrac{1}{2} \max(|s_i(m, n) + 3(j - 1)k_i + p_i(n + j - 1)| + p_i(n + j - 1)), \tag{23}$$

where $p_i(n)$ is the same as in (18) satisfying (19) and the additional condition (20) is also necessary. If $p_i \equiv 0$ for any i, this solution becomes the normal N-soliton solution. We consider it as an ultradiscrete analog to the discrete solution (15) though we cannot show their direct relation by the ultradiscretization. Note that some ultradiscrete solutions in the UP form are also obtained for other ultradiscrete soliton equations [3, 6]. Moreover, we show in section 4 that the perturbation form (18) and the UP form (23) are equivalent to each other and give the same x_n^m to (7) for $M = 2$.

3. Time evolution of periodic phase soliton

In this section, we demonstrate the time evolution of PPS solutions. The solution x_n^m to the uhLV equation (7) for $M = 2$ is obtained from f_n^m to (17) through the transformation (8) for $M = 2$, that is,

$$x_n^m = f_{n-2}^m + f_{n+3}^{m+1} - f_n^m - f_{n+1}^{m+1}. \tag{24}$$

We use the perturbation form (18) to demonstrate. If we assume $p_i \equiv 0$, it becomes the normal soliton solution. Before demonstrating PPS solutions, we demonstrate soliton solutions for comparison. Figure 1 shows 1-soliton solution ($N = 1$ and $p_1 \equiv 0$). Figure 1(*a*) shows the value on the integer lattice points (n, m) and '.' denotes the value 0. Using this representation, the 1-soliton seems to change its shape periodically as time proceeds.

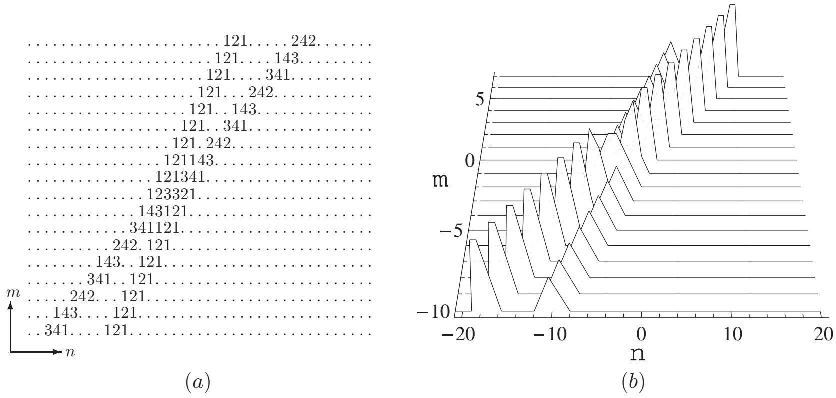

Figure 2. Plot of the 2-soliton solution for $N = 2$, $k_1 = 3$ ($\omega_1 = 5$), $c_1 = 0$, $p_1 \equiv 0$, $k_2 = 1$ ($\omega_1 = 1$), $c_2 = 0$ and $p_2 \equiv 0$. (*a*) Values on lattice points, (*b*) continuous profile for n.

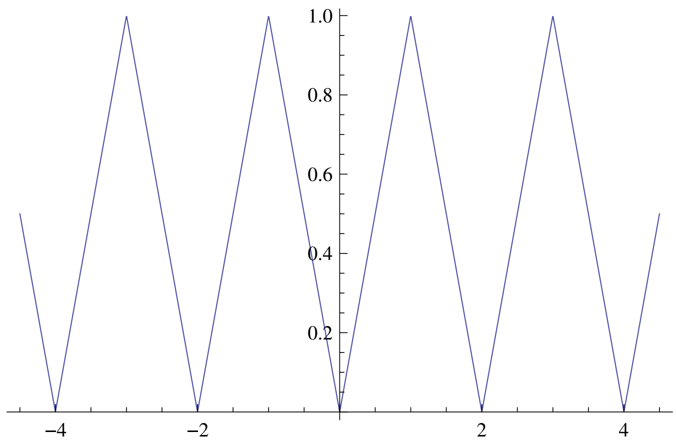

Figure 3. Plot of $p(n)$ defined by (25).

However, we can observe from the continuous profile of solution as shown in figure 1(*b*) that 1-soliton does not change its shape during propagation. Similarly, each soliton of the 2-soliton solution propagates stably without changing its shape except for the interaction period. Figure 2 shows an example of the 2-soliton solution.

We use a function $p(n)$ defined by

$$p(n) = \min(\mathrm{mod}(n, 2), 2 - \mathrm{mod}(n, 2)) \tag{25}$$

to construct the PPS solutions. The graph of this $p(n)$ is shown in figure 3.

The 1-PPS solution is described by

$$f_n^m = \max(0, \omega_1 m - k_1 n + c_1 + p_1(n)). \tag{26}$$

Figure 4 shows the solution x_n^m for $N = 1$, $k_1 = 1$ ($\omega_1 = 1$), $c_1 = 0$ and $p_1(n) = p(n)$, and figure 5 for $N = 1$, $k_1 = 3$ ($\omega_1 = 5$), $c_1 = 0$ and $p_1(n) = 3p(n)$. In both figures, the shape of solitary wave changes periodically. Figures 4(*a*) and 5(*a*) show the values of the solution on the integer lattice points. Since all parameters are integer, the values are always integer.

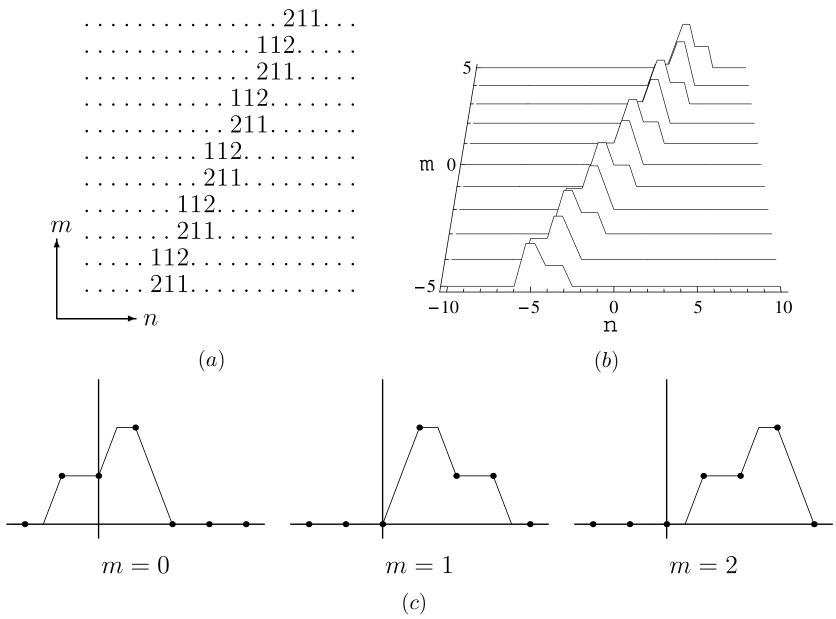

Figure 4. Solution x_n^m for $N = 1$, $k_1 = 1$ $(\omega_1 = 1)$, $c_1 = 0$ and $p_1(n) = p(n)$. (*a*) Values on lattice points, (*b*) continuous profile for n, (*c*) detailed profile at certain time steps.

Figures 4(*b*) and 5(*b*) show the continuous profile of the solution. These figures also indicate the periodic change of profile. It is clear that the PPS solution is of a different type from the soliton solution without periodicity as shown in figure 1. Figures 4(*c*) and 5(*c*) show the detailed profile of solutions at certain time steps.

The 2-PPS solution is described by

$$f_n^m = \max(0, s_1(m,n) + p_1(n), s_2(m,n) + p_2(n), \\ s_1(m,n) + s_2(m,n) - 3k_2 + p_1(n) + p_2(n+1)), \tag{27}$$

where $s_i(m,n) = \omega_i m - k_i n + c_i$. Figure 6 shows the solution in the case of $N = 2$, $k_1 = 3$ $(\omega_1 = 5)$, $c_1 = 0$, $p_1(n) = 3p(n)$, $k_2 = 1$ $(\omega_1 = 1)$, $c_2 = 0$ and $p_2(n) = p(n)$, and the larger PPS passes through the smaller. The larger PPS changes its shape after the interaction due to the phase shift though the smaller does not change.

The solution of a couple of a PPS and a soliton is given by the case of $p_1(n) \equiv 0$ and $p_2(n) \neq 0$ or of $p_1(n) \neq 0$ and $p_2(n) \equiv 0$. Figure 7 shows the solution in the case of $N = 2$, $k_1 = 3$ $(\omega_1 = 5)$, $c_1 = 0$, $p_1(n) \equiv 0$, $k_2 = 1$ $(\omega_2 = 1)$, $c_2 = 0$ and $p_2(n) = p(n)$, and the larger soliton passes through the smaller PPS. In this case, the traveling wave behaving like a PPS is transformed into a soliton without a periodic variation after the interaction. However, such a transformation does not always occur and it depends on the parameters of solution. Figure 8 shows the solution in the case of $N = 2$, $k_1 = 3$ $(\omega_1 = 5)$, $c_1 = 0$, $p_1(n) = 2p(n)$, $k_2 = 1$ $(\omega_2 = 1)$, $c_2 = 0$ and $p_2(n) \equiv 0$, and the larger PPS passes through the smaller soliton. In this case, the transformation of waves does not occur.

4. Outline of proof on the solution

In this section, we give the outline of proof showing that solutions (18) and (23) satisfy the bilinear equation (17). First, we show that solution (23) is equivalent to solution (18). Second,

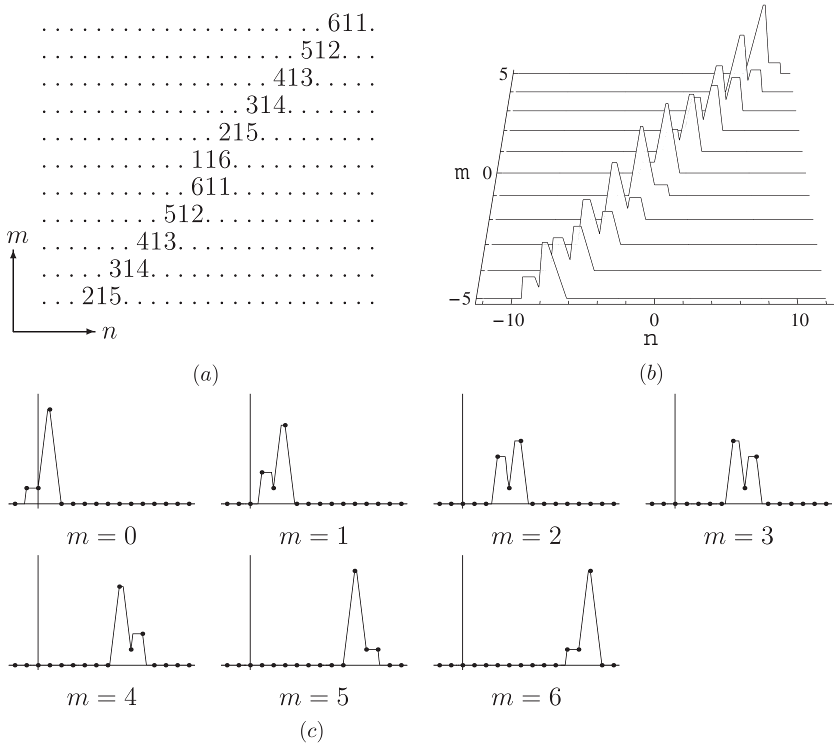

Figure 5. Solution x_n^m for $N = 1$, $k_1 = 3$ ($\omega_1 = 5$), $c_1 = 0$ and $p_1(n) = 3p(n)$. (*a*) Values on lattice points, (*b*) continuous profile for *n*, (*c*) detailed profile at certain time steps.

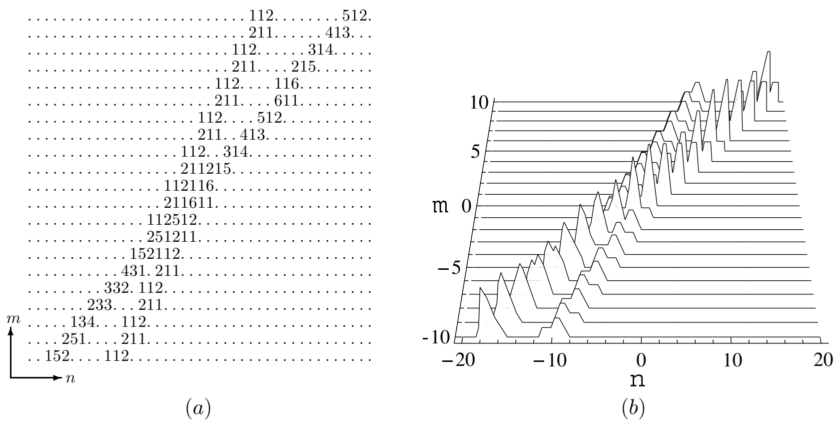

Figure 6. Two PPS solution for $N = 2$, $k_1 = 3$ ($\omega_1 = 5$), $c_1 = 0$, $p_1(n) = 3p(n)$, $k_2 = 1$ ($\omega_1 = 1$), $c_2 = 0$ and $p_2(n) = p(n)$. (*a*) Values on lattice points, (*b*) continuous profile.

we show that solution (23) satisfies the bilinear equation (17). Since the proof is very long, we only show the outline here.

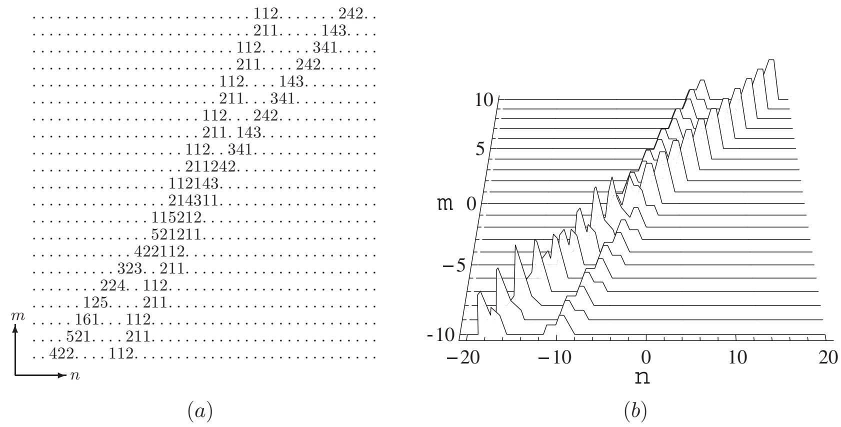

Figure 7. Interaction of a PPS and a soliton for $N = 2$, $k_1 = 3$ ($\omega_1 = 5$), $c_1 = 0$, $p_1(n) \equiv 0$, $k_2 = 1$ ($\omega_2 = 1$), $c_2 = 0$ and $p_2(n) = p(n)$. (*a*) Values on lattice points, (*b*) continuous profile.

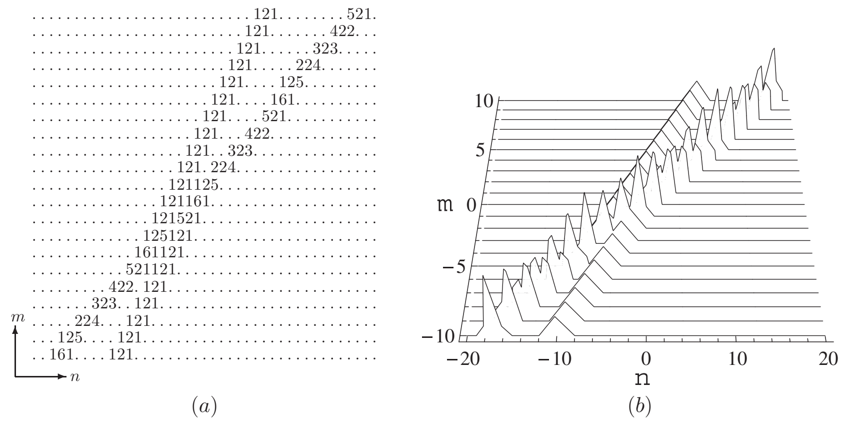

Figure 8. Interaction of a PPS and a soliton for $N = 2$, $k_1 = 3$ ($\omega_1 = 5$), $c_1 = 0$, $p_1(n) = 2p(n)$, $k_2 = 1$ ($\omega_2 = 1$), $c_2 = 0$ and $p_2(n) \equiv 0$. (*a*) Values on lattice points, (*b*) continuous profile.

We may assume $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_N$ for the solution f_n^m of (23) in the UP form without loss of generality. Since $|x| = \max(x, -x)$, we can rewrite f_n^m by

$$f_n^m = \frac{1}{2} \max_{\sigma \in \{1,-1\}^N} A_n^m(\sigma), \tag{28}$$

where $A_n^m(\sigma)$ is an $N \times N$ matrix of which (i, j) element $A_n^m(\sigma)_{ij}$ is given by

$$A_n^m(\sigma)_{ij} = \sigma_i(s_i + 3(j-1)k_i + p_i(n+j-1)) + p_i(n+j-1), \tag{29}$$

and $\sigma = \{\sigma_1, \ldots, \sigma_N\} \in \{-1, 1\}^N$.

Assume that $\sigma = \{\sigma_1, \ldots, \sigma_N\}$ is given and r denotes the number of -1 included in σ. Then we introduce $\pi_i(\sigma)$ ($1 \leqslant \pi_i \leqslant N$) satisfying $\sigma_{\pi_1} = \cdots = \sigma_{\pi_r} = -1$ ($\pi_1 < \cdots < \pi_r$) and $\sigma_{\pi_{r+1}} = \cdots = \sigma_{\pi_N} = +1$ ($\pi_{r+1} > \cdots > \pi_N$). For example, if $N = 6$ and $\sigma = \{1, 1, -1, -1, 1, -1\}$, then we obtain $\pi_1 = 3$, $\pi_2 = 4$, $\pi_3 = 6$, $\pi_4 = 5$, $\pi_5 = 2$, $\pi_6 = 1$.

Moreover, we can obtain the following relation in the case of $i_1 < i_2$ and $j_1 < j_2$:

$$A_n^m(\sigma)_{i_1j_1} + A_n^m(\sigma)_{i_2j_2} - A_n^m(\sigma)_{i_2j_1} - A_n^m(\sigma)_{i_1j_2}$$
$$= \begin{cases} 3(j_2-j_1)\big(k_{i_2}-k_{i_1}\big) + 2\big(p_{i_1}(n+j_1-1) \\ \quad - p_{i_1}(n+j_2-1) - p_{i_2}(n+j_1-1) \\ \quad + p_{i_2}(n+j_2-1)\big) \leqslant 0 & (\text{if } (\sigma_{i_1},\sigma_{i_2}) = (+1,+1)), \\ -3(j_2-j_1)\big(k_{i_2}+k_{i_1}\big) + 2(p_{i_1}(n+j_1-1) \\ \quad - p_{i_1}(n+j_2-1)) \leqslant 0 & (\text{if } (\sigma_{i_1},\sigma_{i_2}) = (+1,-1)), \\ 3(j_2-j_1)\big(k_{i_2}+k_{i_1}\big) - 2\big(p_{i_2}(n+j_1-1) \\ \quad - p_{i_2}(n+j_2-1)\big) \geqslant 0 & (\text{if } (\sigma_{i_1},\sigma_{i_2}) = (-1,+1)), \\ -3(j_2-j_1)\big(k_{i_2}-k_{i_1}\big) \geqslant 0 & (\text{if } (\sigma_{i_1},\sigma_{i_2}) = (-1,-1)). \end{cases} \tag{30}$$

We use the additional condition (20) to derive the above evaluation. Then we can obtain

$$\max A_n^m(\sigma) = A_n^m(\sigma)_{\pi_1 1} + A_n^m(\sigma)_{\pi_2 2} + \cdots + A_n^m(\sigma)_{\pi_N N}. \tag{31}$$

Let us rewrite the above equation by

$$\max A_n^m(\sigma) = A_n^m(\sigma)_{1\pi'_1} + A_n^m(\sigma)_{2\pi'_2} + \cdots + A_n^m(\sigma)_{N\pi'_N}, \tag{32}$$

introducing $\pi'_i(\sigma)$. Then we obtain

$$\pi'_i(\sigma) - 1 = \frac{1+\sigma_i}{2}\left(N - \sum_{j=1}^{i}\frac{1+\sigma_j}{2}\right) + \frac{1-\sigma_i}{2}\sum_{j=1}^{i-1}\frac{1-\sigma_j}{2}. \tag{33}$$

Using these results, the solution f_n^m of (28) is described by

$$f_n^m = \max_{\sigma\in\{1,-1\}^N}\left(\sum_{i=1}^N \sigma_i(s_i + 3(\pi'_i(\sigma)-1)k_i) + \sum_{i=1}^N (1+\sigma_i)p_i(n+\pi'_i(\sigma)-1)\right). \tag{34}$$

This leads to the perturbation form (18) using the transformation of σ_i to μ_i, that is, $\sigma_i = 2\mu_i - 1$. Note that (18) and (23) are different by a linear term of m and n but this term disappears in the rhs of (24). So far, we show the outline of the proof on the equivalence of (18) and (23).

Next we show that f_n^m of (23) satisfies the bilinear equation (17). Three terms, $f_{n+1}^m + f_n^{m+1}$, $f_n^m + f_{n+1}^{m+1}$ and $f_{n-2}^m + f_{n+3}^{m+1}$, are included in (17). They follow the form $f_{n+a}^m + f_{n+b}^{m+1}$ given by

$$f_{n+a}^m + f_{n+b}^{m+1} = \frac{1}{2}\max_{\sigma,\sigma'\in\{-1,1\}^N}\big(\max A_{n+a}^m(\sigma) + \max A_{n+b}^{m+1}(\sigma')\big). \tag{35}$$

If we define λ_i $(1 \leqslant i \leqslant N)$ by $\lambda_i = \sigma_i + \sigma'_i$, λ_i is equal to 2 $(\sigma_i = \sigma'_i = 1)$, 0 $(\sigma_i = 1, \sigma'_i = -1$ or $\sigma_i = -1, \sigma'_i = 1)$ and -2 $(\sigma_i = \sigma'_i = -1)$. Moreover, let us define $F_n^m(a,b,\lambda)$ by

$$F_n^m(a,b,\lambda) = \max_{\lambda=\sigma+\sigma'}\big(\max A_{n+a}^m(\sigma) + \max A_{n+b}^{m+1}(\sigma')\big), \tag{36}$$

where $\lambda = \{\lambda_1,\ldots,\lambda_N\} = \sigma + \sigma' = \{\sigma_1+\sigma'_1,\ldots,\sigma_N+\sigma'_N\}$. We can prove the following equation holds for any λ:

$$F_n^m(1,0,\lambda) = \max\big(F_n^m(0,1,\lambda), F_n^m(-2,3,\lambda) - 2\big). \tag{37}$$

Then we have

$$\max_{\lambda\in\{2,0,-2\}^N} F_n^m(1,0,\lambda) = \max(\max_{\lambda\in\{2,0,-2\}^N} F_n^m(0,1,\lambda), \max_{\lambda\in\{2,0,-2\}^N} F_n^m(-2,3,\lambda) - 2). \tag{38}$$

This means f_n^m of (23) satisfies the bilinear equation (17).

5. Concluding remarks

We propose a new type of solution to the ultradiscrete hungry Lotka–Volterra (uhLV) equation. The periodic phase is introduced into the phase term of soliton. Then the soliton becomes a traveling wave showing a periodic variation. Such a wave is referred as a periodic phase soliton (PPS) in this paper. Since a PPS reduces to a soliton when its periodic phase is set to zero, the PPS solution covers solitons.

It has two equivalent forms of solutions: one is the perturbation form and the other is the ultradiscrete permanent form. We show the concrete demonstration on the solutions in the case of $M = 2$. The interaction among PPSs and solitons is similar to the soliton interaction. However, the interaction gives a complicated phenomenon. For example, we show that a traveling wave behaving like a PPS becomes a soliton after the interaction in the specific case. The detailed analysis about such phenomena is necessary and it is one of the future problems.

We prove that the solution satisfies the ultradiscrete bilinear equation of the uhLV equation in the case of $M = 2$. Since the proof is long, we only give the outline. About the continuous or the discrete soliton equations, we can show that a compact and systematic proof using some formulas on determinants related to the Plücker relations. However, we have not yet obtained the ultradiscrete version of such formulas. It is another future problem to find the formulas and make the systematic proof on the PPS solutions.

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